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Optimizing the Induced Drag

Figure 1: *Wingtip vortex and downwash of an airplane flying in the clouds*

1 Abstract

The drag of a wing can be broken down into two parts, the parasitic drag and the induced drag. The parasitic drag consists of the skin friction and of the pressure drag. There is another form of drag called wave drag, however due to low subsonic speed of this paper, these will not be considered. The induced drag is a drag that is produced when lift is being produced. This drag due to lift is quite big, being approximately 40% of the total drag of a typical transport plane at cruise conditions [1]. Therefore, it is important to try to minimize the induced drag. This research looks into a theoretical analysis of induced drag using lifting line theory to calculate the induced drag. This way, it was found what factors influence the induced drag and how it could be reduced.

2 Introduction

Induced drag or lift-induced drag exists for all lifting surfaces. It is created by wing tip vortexes which in turn create a down-wash over and behind a wing. The induced drag is related to the lift distribution on a wing and can thus be altered by using different wing shapes, profiles, airfoils and more.

Induced drag is an important part of aircraft dynamics since it has influence on multiple aspects of the aircraft. One of these is the extra drag created by the induced drag. This extra drag will result in more fuel used when flying. When the induced drag is reduced the cost of air travel will be lowered, and secondly the effect on the climate can be reduced. The down-wash which creates the induced drag also has an influence on air traffic behind an aircraft. This influence on trailing aircraft has an impact on the frequency of take-off and landing aircraft at an airport. Therefore, when being able to minimize the down-wash of an aircraft and in turn lowering the induced drag could result in higher take-off and landing frequencies of aircraft.

In order to say something about induced drag, a way of calculating the induced drag is needed. Therefore this paper aims to show how to calculate the induced drag using lifting line theory, and how this theory can be used to calculate the effect of winglets on the induced drag.

2.1 Induced Drag

Induced drag on wings occurs due to the presence of lift. The lift of an aircraft is partially created by the pressure difference between the top and bottom side of the wing. Where the bottom is an area of high pressure and the top an area of lower pressure, resulting in a force wanting to push up on the wing, see figure 2.

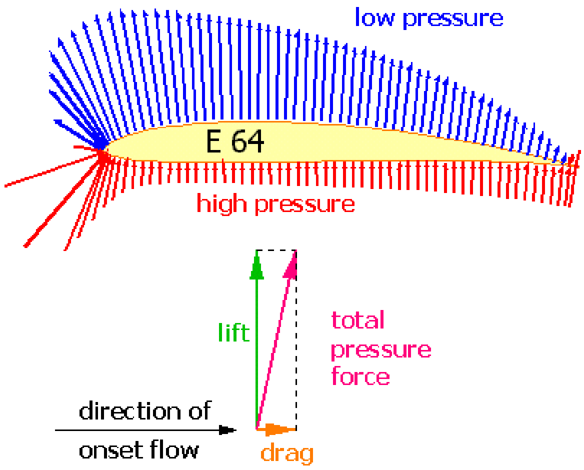

Figure 2: *Pressure distribution on wing section [2]*

Because the length of wings is finite, there exists a leakage at the wing tip. Here the air leaks from the high pressure bottom side of the wing to the lower pressure top side of the wing. This leakage generates a vortex at the wingtips which trails behind the aircraft.

This vortex is created at either end of a wing, and the affect of the two vortices at the wingtips combined results in down-wash across the whole wing (figure 3), which has an effect on the lift and drag of the wing.

The effect of down-wash on the lift and drag of a wing is shown in figure 4, where the down-wash ($-w_{wake}$) adds a vertical component to the incoming airflow (V_∞). This results in an induced angle (a_i) of the airflow resulting in an effective airflow (V_{eff}) acting on the wing. This effective airflow over the wing alters the angle of attack off the wing, this is called the effective angle of attack (a_{eff}). The relation of the effective angle of attack can be seen in equation 1 below.

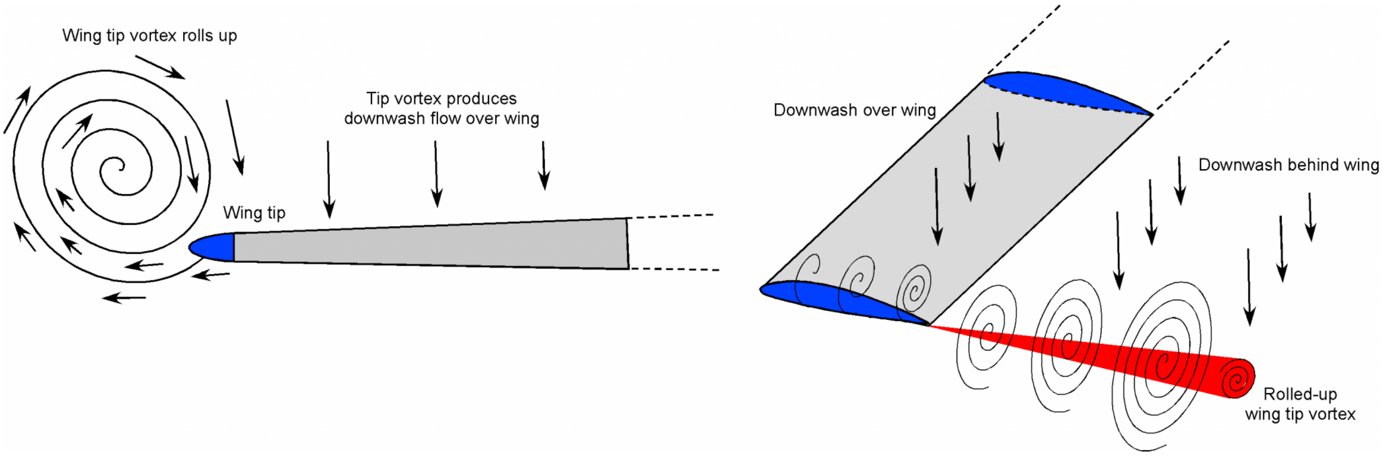

Figure 3: *Wing vortex and down-wash on a wing [3]*

$$a_{eff} = a + a_{aero} - a_i \tag{1}$$

where a is the angle of attack, defined as the angle between the zero lift line and the x-axis, and a_aero is the spanwise twist of the wing, and a_i is the induced angle resulting from the downwash.

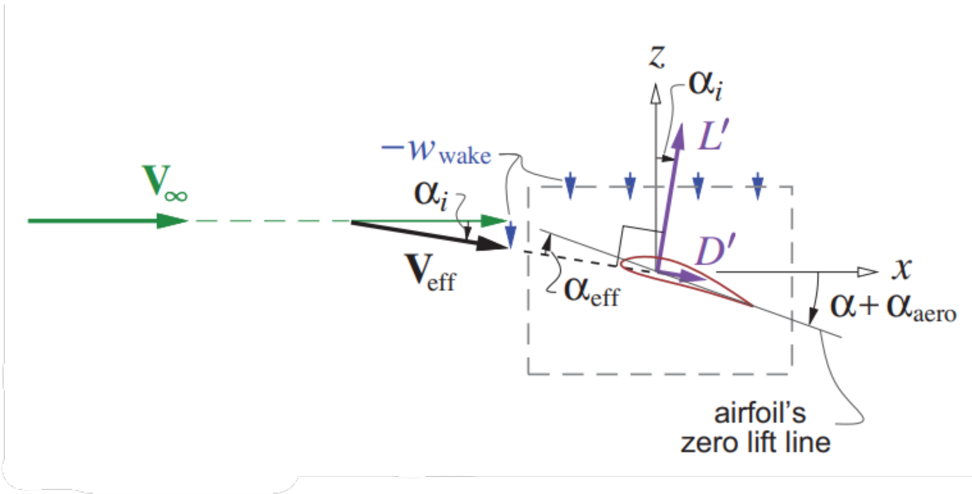

Figure 4: *Forces on finite wing with wake, showing induced drag [4]*

This induced angle rotates the lift to the rear of the wing (L'). This creates and additional drag force in the x-direction when decomposing the new lift force L'. This drag force is called the induced drag and is related to the induced angle as follows.

$$D_i = \int L' a_i \, dy \tag{2}$$

Where it can be seen that the total induced drag of a wing is the summation of induced drag of all wingsection along the span of the wing. Using lifting line theory, in the next sections of this paper, an equation for the induced angle will be derived.

3 Prandtl's Lifting Line Theory

Prandtl's lifting line theory was invented by Prandtl in 1911 and is still widely used nowadays. It can effectively be used to compute the induced drag of a certain finite wing. As explained earlier, a lift producing finite wing creates vortices at the tips. For now a finite flat wing of span s will be assumed, with the middle of the wing at the origin. The y axis points in the direction of the span.
The idea of the lifting line theory is to model these wingtip vortices as two vortex lines leaving from the tips and completely leave out the wing. However, Helmholtz second theorem states: *'a vortex line cannot end in a fluid, it must extend to the boundaries of the fluid or form a closed path'* [5]. Therefore, these vortex lines are modeled to be semi-infinitely long, away from the wing and connected at the trailing edge of the wing, by another vortex line segment, resulting in a so called horseshoe vortex. The resulting velocity induced by the vortices can be calculated using Biot-Savart law as shown in the equation below [6]:

$$V = \frac{\Gamma}{4\pi r} \tag{3}$$

Where r is the distance from the vortex. From this equation it is clear that when being very close to the horse shoe vortex, this induced velocity blows up, going to infinity. This is a problem and should be avoided. Therefore, instead of putting one horseshoe vortex at the back of the foil, more horseshoe vortices will be put, as shown in the figure below.

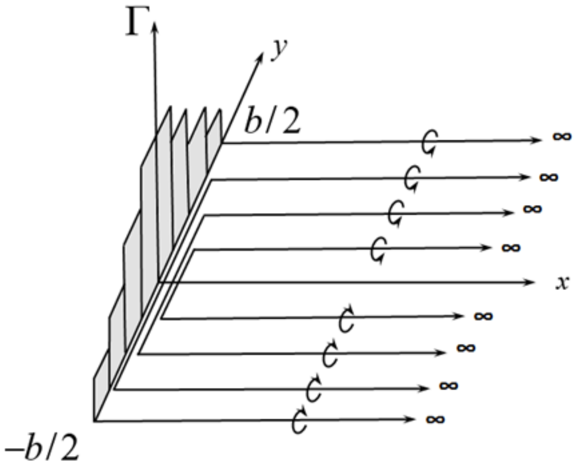

Figure 5: *Illustration of multiple horseshoe vortices at the trailing edge of the wing, together with the circulation distribution*

In this figure, b is used instead of s for the span. It can be seen that when introducing multiple horseshoe vortices all with a certain circulation, the total circulation increases towards the middle and decreases towards the edges. This is because in the middle all horseshoe vortices overlap while on the edges there is only the contribution of one horseshoe vortex. From this it can be seen that if there are infinite amount of horseshoe vortices at the back of the wing, the velocity at the edges will go to zero and thus the problem of the velocity going to infinity at the tips is prevented. Each horseshoe vortex has a circulation $d\Gamma$ occupying a space along the span dz. Now, the induced velocity at a point along the span (y_0) caused by the vortex at y, can be calculated by using the Biot-Savart law once more, as:

$$dv = -\frac{\left(\frac{d\Gamma}{dy}\right) dy}{4\pi(y_0 - y)} \tag{4}$$

In this equation, $\frac{d\Gamma}{dy}$ is the infinitesimal change in circulation with respect to a change of the spanwise coordinate. This is then multiplied with the spanwise change dy in order to find the circulation of the vortex. In order to add the effect of all horseshoe vortices, this equation should be integrated over the whole span of the wing, resulting in:

$$v = -\frac{1}{4\pi}\int_{-s/2}^{s/2}\frac{\left(\frac{d\Gamma}{dy}\right)}{y_0 - y}dy \tag{5}$$

As explained earlier, this vertical velocity called downwash induces an angle on the wing. Figure 4 shows the triangle consisting of the free stream velocity, the induced angle and the wake. It can be seen that the induced angle is equal to:

$$\alpha_i = \arctan\left(\frac{-v}{u_\infty}\right) \quad \rightarrow \quad \alpha_i \approx \frac{-v}{u_\infty} \tag{6}$$

Where u_∞ is the free stream velocity. When considering small angles, the arctan can be ommitted as shown in the equation. The previously calculated downwash can be filled into the equation for the induced angle, resulting in:

$$\alpha_i = \frac{1}{4\pi u_\infty}\int_{-s/2}^{s/2}\frac{\left(\frac{d\Gamma}{dy}\right)}{y_0 - y}dy \tag{7}$$

Next up, with the induced angle, the effective angle of attack can be calculated. It depends on the angle of attack of the wing, the induced angle and on the twist distribution of the wing.

$$\alpha_{eff} = \alpha - \alpha_i \tag{8}$$

α is a combination of both the angle of attack and the twist distribution and can thus be a function of y. From thin airfoil theory, the following relation is known for the lift coefficient as a function of the angle of attack [7]:

$$\frac{dC_l}{d\alpha} = 2\pi \quad \rightarrow \quad C_l = 2\pi(\alpha_{eff} - \alpha_{L=0}) \tag{9}$$

Where the zero lift angle is usually known for the airfoil profile used. Now that the effective angle of attack of the wing is known, the lift can be calculated using the lift equation. This lift can be related to the circulation with the Kutta Joukouwsky theorem. This theorem states that the lift distribution can be related to the circulation distribution. The lift equation and the Kutta Joukouwsky theorem are shown in the equation below. The equations are combined to find an expression for C_l [8]:

$$\begin{aligned} L'_{(y)} &= \frac{1}{2}\rho u_\infty^2 c C_l \\ L'_{(y)} &= \rho u_\infty \Gamma_{(y)} \quad \rightarrow \quad C_l = \frac{2\Gamma}{u_\infty c} \end{aligned} \tag{10}$$

Where c is the chord of the wing which can be a function of y. This final expression for C_l can be set equal to the equation for the lift coefficient found by thin airfoil theory as shown in Equation 9, resulting in:

$$\alpha_{eff} = \frac{\Gamma}{\pi u_\infty c} + \alpha_{L=0} \tag{11}$$

Now, an expression has been found for the effective angle of attack as a function of Γ. Previously the equation for the induced angle of attack was found as a function of Γ. This means that the angle of attack of the wing α, can be expressed fully as a function of the circulation Γ. This is great because the angle α is known, as it just depends on the cruise conditions and twist distribution of the wing. Combining the equations gives:

$$\boxed{\alpha = \alpha_{eff} + \alpha_i = \frac{\Gamma}{\pi u_\infty c} + \alpha_{L=0} + \frac{1}{4\pi u_\infty}\int_{-s/2}^{s/2}\frac{\left(\frac{d\Gamma}{dy}\right)}{y_0 - y}dy} \tag{12}$$

This is the fundamental equation of the lifting line theory. The only unknown is Γ, so the equation can be solved. However, it is a differential equation which is hard to solve. However, if the circulation distribution is known, it can be used to calculate the lift distribution using the Kutta-Joukouwsky theorem. This can then be used to calculated the induced drag using Equation 2.

3.1 Lifting Line Theory for Elliptical Lift Distribution

What can be seen in lifting line theory, is that the circulation is the only unknown in the previously described equations. When the circulation of a certain wing is known, it is possible to calculate the lift distribution (13), lift (14) and induced drag (15).

$$L'(y) = \rho u_\infty \Gamma(y) \tag{13}$$

$$L = \rho u_\infty \int_{-s/2}^{s/2} \Gamma(y)\, dy = \int_{-y/2}^{y/2} L'\, dy \tag{14}$$

$$D_i = \rho u_\infty \int_{-s/2}^{s/2} a_i\, dy \tag{15}$$

As an example of the circulation Γ an elliptical distribution is used to find expressions of these equations

$$\text{Ellicptical distrubution: } \Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{y}{s/2}\right)} \tag{16}$$

Where Γ_0 is the peak circulation at the middle of the wing. This distribution can then be used to find a the lift distribution along the wing.

$$L'(y) = \rho u_\infty \Gamma_0 \sqrt{1 - \left(\frac{y}{s/2}\right)} \tag{17}$$

The lift distribution is then can be used to calculate the lift of the wing. Where the lift distribution is integrated along the span of the wing giving the following.

$$L = \int_{-s/2}^{s/2} L'\, dy = \rho u_\infty \Gamma_0 s \frac{\pi}{4} \tag{18}$$

Then it is possible to equate this equation to the definition of the lift coefficient. This being.

$$L = C_L \frac{1}{2} \rho u_\infty^2 A \tag{19}$$

This can then give us an expression for Γ_0 giving the following.

$$\Gamma_0 = \frac{2 u_\infty A C_L}{s\pi} \tag{20}$$

To use the lifting line equations the vorticity distribution is needed. This can be described as follows.

$$\gamma(y') = \frac{d\Gamma}{dy}(y') = -\Gamma_0 \frac{4y}{s^2} \frac{1}{\sqrt{1 - \left(\frac{2y}{s}\right)^2}} \tag{21}$$

For easier computations a cosine transformation is performed using the following defention.

$$y = \frac{s}{2} \cos\theta \text{ where } \theta = [\pi, 0] \tag{22}$$

The expressions above can be used to find the induced velocity over the wing using the following formula.

$$v(y) = -\frac{1}{4\pi}\int_{-s/2}^{s/2}\frac{\gamma(y')dy'}{(y-y')} \tag{23}$$

Filling in the expressions above and the cosine transform gives

$$v(\theta) = -\frac{\Gamma_0}{2\pi s}\int_{\pi}^{0}\frac{\cos\theta'}{\cos\theta-\cos\theta'}d\theta' \tag{24}$$

From thin airfoil theory it can be seen that this equation can be reduced to:

$$v(\theta) = -\frac{\Gamma_0}{2s} \tag{25}$$

This shows that the downwash is constant along the whole wing. And thus that an elliptical circulation distribution results in constant downwash along the span of the wing.

Now filling in the known Γ_0 into the equation for downwash gives

$$v = -\frac{\Gamma_0}{2s} = -\frac{V_\infty C_L}{AR\pi} \tag{26}$$

This can be rewritten using the definition of the induced angle

$$\tan\alpha_i = \frac{-v}{V_\infty} = \frac{C_L}{\pi AR} \tag{27}$$

This induced angle can be releated to lift and drag as can be seen in figure 4.

$$\tan\alpha_i = \frac{D_i}{L} \tag{28}$$

rearranging and normalizing

$$C_{Di} = C_L\tan\alpha_i \tag{29}$$

If equation 27 is substituted into an equation for induced drag is found.

$$C_{Di} = \frac{C_L^2}{\pi AR} \tag{30}$$

Where AR is the Aspect Ratio, and this equation shows the induced drag for an elliptical circulation distribution, and is said to be the most efficient distribution to minimize induced drag. This equation can be made more general by integrating and efficiency factor e where $e = 1$ for elliptical lift distributions and $e \leq 1$ for other lift distributions giving:

$$C_{Di} = \frac{C_L^2}{\pi eAR} \tag{31}$$

3.2 Method of Restricted Variations

Another way to find the minimum of the induced drag is to use the method of restricted variations. In order to use this theorem, first the induced drag is rewritten. It can be rewritten using Kutta-Joukouwsky's theorem:

$$L'_{(y)} = \rho u_\infty \Gamma_{(y)} \quad \rightarrow \quad D'_i = \rho u_\infty \alpha_i \Gamma_{(y)} \quad \rightarrow \quad D'_i = \rho v \Gamma \tag{32}$$

Where v is the downwash and the minus sign was ommitted for simplicity. This equation directly relates the circulation and downwash to the induced drag. Lets assume a certain circulation distribution Γ, as shown in the figure below.

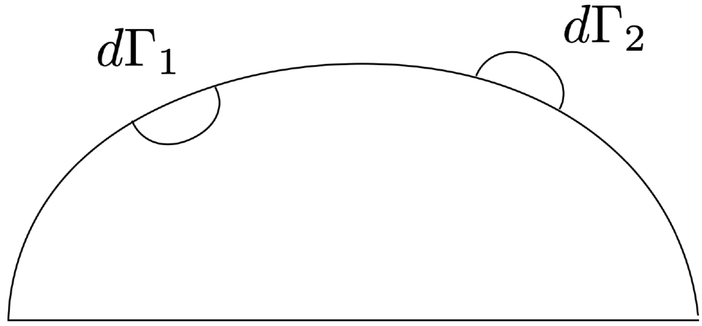

Figure 6: *An arbitrary circulation distribution, together with two infinitesimal changes [9]*

Now imagine that two infinitesimal changes are made to this circulation distribution $d\Gamma_1$ and $d\Gamma_2$. In order to still produce the same amount of lift for this wing, the two changes must net zero, such that the net lift of the wing is still the same, resulting in:

$$d\Gamma_1 + d\Gamma_2 = 0 \tag{33}$$

The change in induced drag because of these two circulation changes can be expressed, using Equation 32 as shown in the following equation:

$$\delta D_i = v_1 d\Gamma_1 + v_2 d\Gamma_2 \tag{34}$$

When the minimum induced drag is reached, the change in induced drag should equal to zero. Combining Equation 33 with Equation 34 shows that the solution is:

$$v_1 = v_2 \tag{35}$$

So, the minimum drag is achieved when the downwash at the two points is equal. However, these points can be chosen arbitrarily along the span of the wing. Therefore, the downwash should be constant along the whole span of the wing. It was earlier shown that an elliptical wing produces a constant downwash along the span, and this result shows that indeed the elliptical lift distribution results in the minimum induced drag.

3.3 Finding the circulation distribution of an arbitrary wing

For arbitrary wing profiles the lift distribution $\Gamma(y)$ might be still unknown. However, it is possible to get an estimate of the circulation distribution givin a wing and its properties.

The circulation distribution of a wing can be found using the following steps:

1. Assume a circulation distribution $\Gamma(y)$ (Elliptical is a good start)
2. From circulation distribution$\Gamma(y)$ solve the induced angle a_i

3. From induced angle a_i solve the effective angle a_{eff}
4. From effective angle a_{eff} estimate C_L using equation $C_L = 2\pi(a_{eff} - a_{L=0})$
5. Using C_L –> calculate lift distribution L' –> find expression for the circulation distribution $\Gamma(y)$ using Kutta Joukouski condition ($L'_{(y)} = \rho u_\infty \Gamma_{(y)}$)
6. Repeat steps 1-5 until with the found circulation distribution $\Gamma(y)$ in step 5 untill $\Gamma(y)$ is converged

This converged circulation distribution $\Gamma(y)$ is the circulation around the given wing. $\Gamma(y)$ can then for instance be used to calculate the induced drag of of a wing, or to evaluate the distribution compared to the elliptical distribution.

4 Method of Restricted Variations for Winglets

Another way to reduce the induced drag is to use winglets, which are devices at the tip of a wing. They are less efficient then increasing the span of the wing but are still important, especially for wings that are span limited or limited on the bending moment at the root of the wing [4]. This section goes into the method of restricted variations for winglets. When looking at a winglet, it can be considered as a small part of a wing that has an dihedral angle as shown in the figure below:

Figure 7: *A sketch of a wing with a winglet at a angle ϕ*

Now just like shown in the previous section, the method of restricted variations can be used to optimize the induced drag of the wing with the winglet. For this approach, the dihedral angle along the entire wing can vary. If the wing has a dihedral angle, then the direction of the circulation changes across this piece of the wing, because the horse shoe vortices are also differently oriented on this piece of the wing. In order to keep it general, the normal wash will now be used instead of the downwash, since the induced angle for a dihedral wing depends on the normal wash. Therefore, the minimum of the induced drag by the method of restricted variations becomes:

$$\delta D_i = 0 \rightarrow V_{n1} d\Gamma_1 + V_{n2} d\Gamma_2 = 0 \tag{36}$$

Now just like before, the change in lift should be zero. For this, only the down component of the circulation is used because that contributes to lift, resulting in:

$$\delta L = 0 \rightarrow d\Gamma_1 \cos\phi_1 + d\Gamma_2 \cos\phi_2 = 0 \tag{37}$$

Combining the two equations finds the following result for the minimum induced drag:

$$\boxed{V_n = v_0 \cos\phi} \tag{38}$$

So, this shows that the local normal wash should be equal to some constant multplied by the local cosine. This equation tells something important about winglets. Let's apply this equation to a winglet at 90° dihedral.

4.1 Vertical winglet (90°)

When looking at the back of a wing, the vortex induces a sidewash onto the winglet. This can be seen in the figure below.

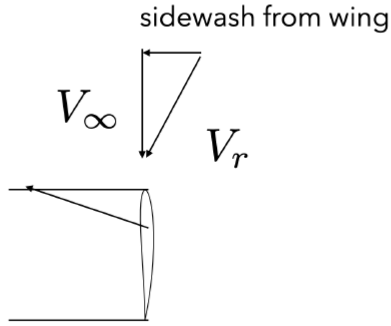

Figure 8: *Sidewash induced by the wing onto the winglet and the lift direction [9]*

The sidewash caused by the wing, combined with the freestream velocity leads to a angle of attack on the winglet. The lift is perpendicular to this total instream velocity V_r. Therefore, it produces a bit of thrust, meaning that the net induced drag is lowered. However, according to Equation 38, this is not the optimal loading for a winglet. When filling in a winglet with a dihedral angle of 90°, the local net normal wash should equal 0. So, it would be more optimal if the winglet creates a sidewash in the other direction of the sidewash of the wing, such that there is no net normal wash. This way, the winglet does not produce thrust. However, the winglets vortices cause an upwash on the main wing (mainly near the tip) that reduces the induced drag of the wing.

5 Conluding remarks

Concluding, this research has looked into the optimization of the induced drag for a finite wing. Tools that were used were lifting line theory and the method of restricted variations. Through lifting line theory, it was found that the most optimal lift distribution is an elliptical lift distribution, where the downwash is constant along the span. Moreover, the lifting line result showed that maximizing the aspect ratio results in a smaller induced drag. When the aspect ratio is limited, winglets can be used to further decrease the induced drag. The method of restricted variations has shown that a vertical winglet is the most efficient when it does not produce a net thrust. In general, however, stating the effect of winglets on induced drag is difficult and is probably better done using tools like CFD software or a wind tunnel measurement.

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